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A new indirect formulation to address the non-uniqueness problem in acoustic BEM

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ABSTRACT

In this paper we introduce a new indirect Boundary Element (BE) formulation for acoustics, and discuss some of its advantages over conventional formulations.

Conventional BE formulations are limited in their capability to allow different *types* of boundary conditions on the two sides of a surface. The discontinuous boundary conditions in conventional BE formulations require the same boundary condition type (known pressure, known velocity or known impedance) to be specified on both sides of the surface. Our BE formulation is very general and allows *different boundary condition types* on either side. For example, pressure can be specified on one side, while specifying velocity on the other side. This complete decoupling of the boundary conditions on the two sides allows for greater modeling flexibility.

The non-uniqueness problem is very common in acoustic boundary element methods. In traditional indirect BE formulations, dummy elements with specified impedance are added to the model interior in an ad hoc manner to overcome non-uniqueness. In this paper, we discuss how our new indirect BE formulation uses different boundary condition types to overcome the non-uniqueness problem. An example problem (radiation from a sphere) is presented to demonstrate the solution accuracy over a wide frequency range.

1. INTRODUCTION

Prediction of acoustic radiation from various engineering structures such as automotive components, wind turbines, aircraft and submarine applications is crucial in designing low-noise structures. Numerical solution methods such as Finite Element Method (FEM) and Boundary Element Method (BEM) offer powerful computational tools to model complex industrial structures and boundary conditions. BEM is the most widely used numerical method in acoustics due to its intrinsic advantages in modeling radiation problems. The acoustic integral equations, derived from the linear wave equation, over a domain volume can be recast over the surface of the structure in BEM. This reduces the dimensionality of the problem by one, a 3-D acoustic problem is reduced to 2-D in BEM. Moreover, the radiation condition for an unbounded exterior domain is automatically satisfied in an acoustic Boundary Element (BE) formulation.

This paper deals with a new indirect BE formulation developed to handle variety of discon-

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tinuous boundary conditions. The discontinuous boundary conditions in conventional indirect BE formulations are limited to the same boundary condition type (known pressure, known pressure gradient or known impedance) to be specified on both sides of the boundary. However, our new formulation allows specification of different boundary condition types on either side. For example, known pressure can be specified on one side, while specifying known velocity on the other side. This complete decoupling of the boundary conditions on either side of the boundary offer greater modeling flexibility. All these boundary conditions are successfully implemented in COUSTYX, a general purpose boundary element program for acoustics.

It is well known that boundary integral equations fail to have *unique* bounded solutions when the analysis frequency coincides with some critical frequencies, also called *irregular frequencies*. These irregular frequencies are the resonance frequencies of the interior problem. When we are only interested in the exterior problem, these interior resonances seriously degrade the accuracy of the exterior solution. The non-uniqueness issue can be overcome by simultaneously employing the velocity and unequal impedance boundary condition on the BE model.[1] The resonance effect of the interior solution is significantly diminished by assigning a small value for the impedance on the interior side of the surface and a large value on the exterior side. However, this technique can't be employed to eliminate irregular frequencies for a pressure boundary condition. We propose an alternate general indirect BE formulation which allows *different boundary condition types* on each side of the boundary and these boundary conditions can be employed appropriately to treat irregular frequencies effectively.

2. INDIRECT BOUNDARY ELEMENT FORMULATION

In the Indirect BEM formulation, the solution to the Helmholtz equation is expressed in terms of surface potentials. The jump properties of the surface potentials are used to arrive at integral equations that relate the primary variables – single-layer density σ and double-layer density μ, to the applied boundary conditions. Figure 1 illustrates an acoustic problem with a boundary surface S. The positive (side 1) and negative (side 2) sides of the boundary are conveniently defined to be on the side of the normal and away from the normal respectively.

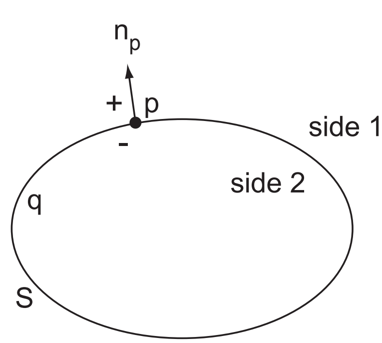


Figure 1: Boundary surface of an acoustic structure.

A. Surface Potentials

In Indirect BEM the surface potentials are pressure fields due to monopole and dipole source distributions on the surface. The single layer potential is the pressure field due to a monopole source distribution on the boundary and is expressed in terms of single layer density σ. The double layer potential is the pressure field due to a dipole source distribution and is expressed in terms of double layer density μ.

The pressure field $\tilde{p}(p)$ at any point p due to the single and double layer potentials is given by the Helmholtz Integral Equation (HIE, $e(p)$) as

$$e(p) = \tilde{p}(p) = -\int_S \sigma(q) G_H(q,p) dS(q) + \int_S \mu(q) \frac{\partial G_H(q,p)}{\partial n_q} dS(q) \tag{1}$$

where $G_H(q,p)$ is the free-space Green's function.

The derivative of the pressure at any point p in the direction n_p is given by the Normal Derivative Integral Equation (NDIE, $f(p)$) as

$$f(p) = \frac{\partial \tilde{p}(p)}{\partial n_p} = -\int_S \sigma(q) \frac{\partial G_H(q,p)}{\partial n_p} dS(q) + \int_S \mu(q) \frac{\partial^2 G_H(q,p)}{\partial n_p \partial n_q} dS(q) \tag{2}$$

The discontinuity of the single and double layer potentials at the boundary S results in the following jump properties.

$$\sigma(p) = \frac{\partial \tilde{p}(p^+)}{\partial n_p} - \frac{\partial \tilde{p}(p^-)}{\partial n_p} \tag{3}$$

$$\mu(p) = \tilde{p}(p^+) - \tilde{p}(p^-) \tag{4}$$

The jump properties of the single layer and double layer potentials at the boundary surface S are important to establish relations between the densities σ, μ and the prescribed boundary conditions on the surface. The HIE and NDIE expressions along with the jump properties are used to derive the integral equations in indirect BE formulation.

B. Boundary Conditions

The indirect BE formulation is derived for a variety of boundary condition types. Boundary conditions are necessary to solve for the unknown surface densities, σ or μ, and compute field pressure at any point. Because of the definition of the primary variables indirect BEM solves both sides of the boundary at the same time. Our new indirect BE formulation handles wide variety of boundary conditions on both sides of the boundary surface. We will discuss some of the most commonly used boundary conditions and derive the integral equations for them.

The boundary conditions we have implemented in our formulation which are relevant to this paper are broadly classified below.[2]

1. Continuous boundary conditions: The boundary conditions employed on both sides of the surface S are of the same type and have the same value. These are the most commonly used boundary conditions. For example, a continuous velocity boundary condition specifies equal velocity on side 1 and side 2 of the surface. The integral equations derived for these boundary conditions either have HIE or NDIE.

2. Discontinuous boundary conditions – Same type: The boundary conditions employed on both sides of the surface S are of the same type but have different values. For example, a discontinuous velocity boundary condition specifies unequal velocities on side 1 and side 2 of the surface. The integral equations either have HIE or NDIE.

3. Discontinuous boundary conditions – Different types: The boundary conditions employed on both sides of the surface S are of different types. The general relation between pressure and its normal derivative on side 1 and side 2 are specified. For example, a discontinuous boundary condition can be applied with a velocity on one side of the surface and pressure on the other. The integral equations have both HIE and NDIE.

Our main contribution lies in the implementation of discontinuous boundary condition with different types of known values on either side of the boundary. These boundary conditions are also effectively used in eliminating *non-uniqueness* in the boundary integral solution.

B.1. Discontinuous Boundary Conditions – Different Types

We implemented this boundary condition by representing the boundary conditions on each side in a very general form. The general relation between pressure and its normal derivative on each side of the surface is specified as follows.

$$\alpha^+(p)\tilde{p}(p^+) + b^+(p)\tilde{p}_n(p^+) = \gamma^+(p) \tag{5}$$

$$\alpha^-(p)\tilde{p}(p^-) + b^-(p)\tilde{p}_n(p^-) = \gamma^-(p) \tag{6}$$

where α, b and γ are complex coefficients relating the pressure $\tilde{p}(p)$ with the normal derivative of the pressure $\tilde{p}_n(p)$ on side 1 $(+)$ and side 2 $(-)$. The single-layer and double-layer densities for discontinuous boundary condition of different types are non-zero. Hence both the pressure and its normal derivative have jumps on traversing the surface.

$$\begin{array}{ll} \tilde{p}(p^+) = e(p) + \frac{\mu(p)}{2}, & \tilde{p}_n(p^+) = f(p) + \frac{\sigma(p)}{2} \\ \tilde{p}(p^-) = e(p) - \frac{\mu(p)}{2}, & \tilde{p}_n(p^-) = f(p) - \frac{\sigma(p)}{2} \end{array} \tag{7}$$

Two coupled boundary integral equations are reduced from the boundary condition relations in Equations 5, 6, and the jump properties in Equation 7. The first of the coupled integral equations that is satisfied at any point p on S is expressed in terms of HIE ($e(p)$),

$$\begin{aligned} -e(p) - \frac{\mu(p)}{2}\left[\frac{\alpha^+(p)b^-(p) + \alpha^-(p)b^+(p)}{\alpha^+(p)b^-(p) - \alpha^-(p)b^+(p)}\right] - \sigma(p)\left[\frac{b^+(p)b^-(p)}{\alpha^+(p)b^-(p) - \alpha^-(p)b^+(p)}\right] \\ - \left[\frac{b^+(p)\gamma^-(p) - \gamma^+(p)b^-(p)}{\alpha^+(p)b^-(p) - \alpha^-(p)b^+(p)}\right] = 0 \quad \forall p \in S \end{aligned} \tag{8}$$

The second of the coupled integral equations that is satisfied at any point p on S is expressed in terms of NDIE ($f(p)$),

$$\begin{aligned} f(p) - \frac{\sigma(p)}{2}\left[\frac{\alpha^+(p)b^-(p) + \alpha^-(p)b^+(p)}{\alpha^+(p)b^-(p) - \alpha^-(p)b^+(p)}\right] - \mu(p)\left[\frac{\alpha^+(p)\alpha^-(p)}{\alpha^+(p)b^-(p) - \alpha^-(p)b^+(p)}\right] \\ - \left[\frac{\alpha^+(p)\gamma^-(p) - \alpha^-(p)\gamma^+(p)}{\alpha^+(p)b^-(p) - \alpha^-(p)b^+(p)}\right] = 0 \quad \forall p \in S \end{aligned} \tag{9}$$

Thus the indirect formulation for this boundary condition type readily employs a combination of HIE and NDIE in its equations, which helps eliminate interior resonance effects on the exterior field solution at irregular frequencies.[3]

The variational approach is used to minimize the residue and obtain the system of equations. The functionals associated with each boundary condition are added together to construct a functional over the entire boundary surface. The surface boundary is then discretized into boundary elements and the system of equations are solved numerically.

3. IRREGULAR FREQUENCIES

The primary variables σ and μ in an indirect BE solution represent both the interior and exterior problem. At resonance frequencies of the interior problem, the large relative difference between the resonance response at the interior field and the solution at the exterior field affects the numerical accuracy of the computed values of primary variables. This affects the accuracy of the exterior field solution in radiation problems at irregular frequencies.

To obtain accurate solutions for a radiation problem at all frequencies we need to eliminate the resonance effects of the interior problem. We employed two different strategies to reduce the effect of interior resonance on the surface solution. They are

1. Method 1 – Discontinuous boundary condition with different types on each side of the surface.
2. Method 2 – Discontinuous boundary condition with the same type but unequal values on each side of the surface. Force the interior side boundary condition to zero.

Method 1 uses the idea that a combination of HIE and NDIE in the system equations effectively overcomes the non-uniqueness.[3] For a single type of continuous boundary condition applied on the surface only one of the HIE or NDIE appears in the system equations. For example, when a continuous velocity boundary condition is specified on the surface only NDIE appears in the system equations allowing the interior resonance effects to influence the exterior solution. However, for a problem with a combination of pressure and velocity boundary conditions, either on different portions of the surface or on different sides of the surface (like discontinuous boundary condition with different types on each side), both HIE and NDIE appear in the system equations. Consider an example radiation problem with velocity boundary condition on the exterior side. The interior resonance effects on the exterior field solution can be eliminated by employing pressure boundary condition on the interior side of the surface. This results in the appearance of both HIE and NDIE in the system equations.

In Method 2, interior resonances are suppressed by forcing zero excitation boundary condition on the interior side of the surface. This method is employed by specifying unequal discontinuous boundary condition of the same type on both sides of the surface and forcing the interior side value of the boundary condition to zero. By introducing zero excitation on the interior side the resonance effects are suppressed.

Each of the two methods described above have their advantages and trade-offs. Method 1, which employs discontinuous boundary conditions of different types, doubles the problem size by introducing additional σ or μ variables on the surface. However, the combination of HIE and NDIE in the formulation guarantees that the exterior field solution is not affected by any interior resonance frequencies. Method 2, which employs unequal discontinuous boundary conditions of the same type, is specially attractive because of the smaller number of unknown primary variables.

To demonstrate the effectiveness of the above methods in eliminating resonance effects at irregular frequencies, we solve a pulsating sphere radiation problem. The exterior sound pressure field from BEM is verified with the analytical solution.

A. Pulsating Sphere Radiation Problem

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wave number at a frequency ω is given as $k = \omega/c$. The sphere is pulsating with a radial velocity $v_r = 1\,\mathrm{m/s}$. The BE mesh of the sphere is shown in Figure 2.

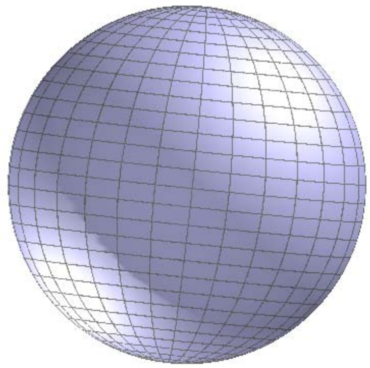


Figure 2: Boundary element mesh for a sphere of unit radius.

The resonance frequencies for a rigid sphere are obtained by solving the corresponding eigenvalue problem. The interior eigenvalue problem, and its solution are described below. At any point inside the sphere, the wave equation must be satisfied.

$$\nabla^2 p + k^2 p = 0$$

On the surface the boundary condition is zero radial velocity.

$$v_r|_{r=a} = 0$$

That means, the radial pressure gradient at the surface must vanish.

$$\left.\frac{\partial p}{\partial r}\right|_{r=a} = 0$$

At some values of k, it is possible to have non-trivial solution for the pressure inside the sphere, even if there is no excitation. These frequencies are the natural frequencies of the interior sphere problem with velocity boundary condition.

A solution of the wave equation valid inside the sphere is

$$p = A\frac{\sin kr}{kr} = Bj_0(kr)$$

j_0 is the spherical Bessel function. This solution is bounded at the origin and has only radial dependence.

$$\left.\frac{\partial p}{\partial r}\right|_{r=a} = 0 = A\left.\frac{k^2 r\cos kr - k\sin kr}{(kr)^2}\right|_{r=a} = \frac{A}{ka^2}\left(ka\cos ka - \sin ka\right)$$

A non-trivial solution ($A \neq 0$), is only possible for values of k that satisfy the equation

$$ka\cos ka - \sin ka = 0$$

or

$$\tan ka = ka$$

Table 1: First few eigenvalues of the sphere interior problem with velocity boundary condition.

n	$k_n a$	$freq(Hz)$
0	0	0
1	4.49	245.30
2	7.73	421.72
3	10.90	595.26

The eigen frequencies of the above solution are obtained from solving $ka\cos(ka) - \sin(ka) = 0$ or $\tan(ka) = ka$. The first few solutions are given in Table 1. At these irregular frequencies the exterior field solutions for the pulsating sphere radiation problem show large errors.

The analytical solution for the radiated pressure p at a distance r from the center of a pulsating sphere is

$$p(r) = \frac{a}{r}\frac{(-ika)}{(1-ika)}Z_o v_r e^{(ik(r-a))} \tag{10}$$

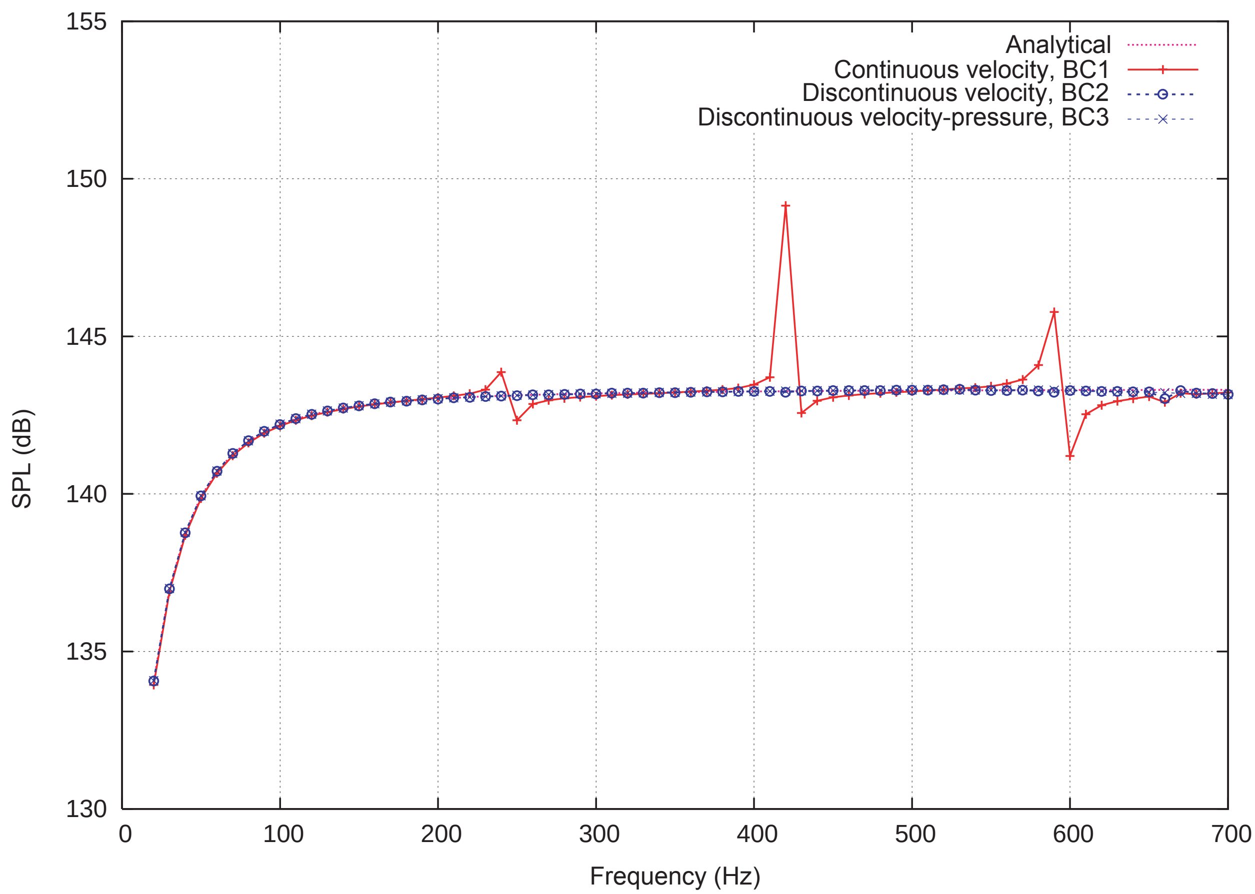


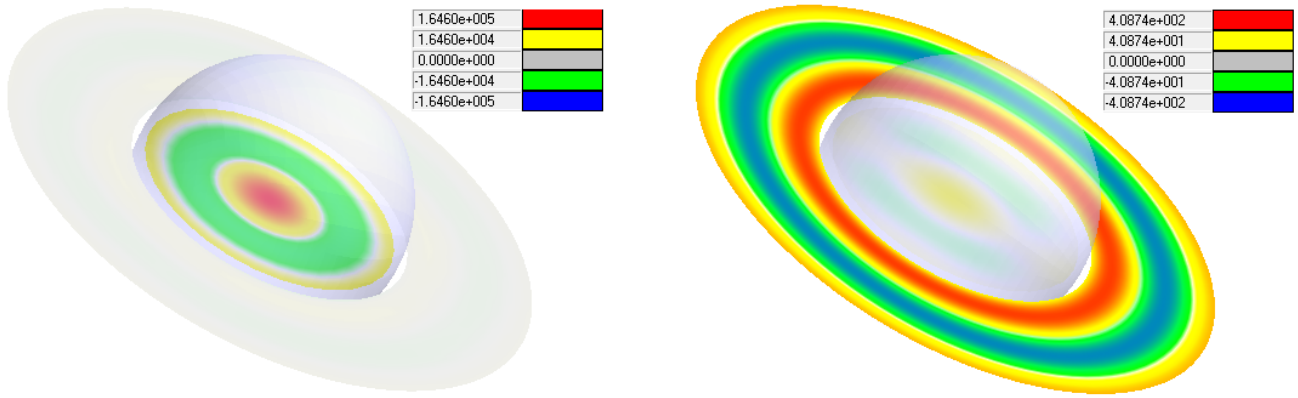
Figure 3: Exterior sound pressure level comparisons on the surface of a sphere with different types of boundary conditions. Reference pressure, $P_{ref} = 20\mu Pa$

Table 2: Different boundary condition types used to solve the pulsating sphere radiation problem. Note that $+$ represents the exterior side, and $-$ represents the interior side of the boundary.

BC1	Continuous normal velocity boundary condition; $v_n^- = v_n^+ = v_r$
BC2	Unequal discontinuous normal velocity boundary condition; $v_n^- = 0$, $v_n^+ = v_r$
BC3	Discontinuous boundary condition with pressure on the inside and normal velocity on the outside of the surface; $p^- = 0$, $v_n^+ = v_r$

The pulsating sphere radiation problem is solved using indirect BEM in three different ways by employing the three different types of boundary conditions described below (refer to Table 2): (1) BC1 – A continuous velocity boundary condition with both the interior and exterior sides having the same unit normal velocity; (2) BC2 – A discontinuous velocity boundary condition with unit normal velocity on the exterior side and zero normal velocity on the interior side; and (3) BC3 – A discontinuous boundary condition with unit normal velocity on the exterior side and zero pressure on the interior side. The solution for the problem with BC1 boundary condition show large errors at the irregular frequencies identified in Table 1. We employ a discontinuous velocity boundary condition (BC2) with zero normal velocity on the interior side and a unit normal velocity specified on the exterior side. This boundary condition forces the excitation to the interior resonance to zero, thus eliminating resonance effects on the exterior solution. The discontinuous boundary condition with zero pressure on the interior side and a unit normal velocity on the exterior side (BC3) completely eliminates resonance effects at all frequencies. Figure 3 shows comparisons of sound pressure levels at an exterior point on the surface obtained from BE solutions to the three different boundary conditions BC1, BC2 and BC3 with the analytical solution. Suppression of resonance effects are clearly demonstrated in the solutions from both the methods in Figure 3. These solutions match well with the exact analytical solution. The non-uniqueness problem in indirect BEM can be resolved by treating the acoustic problem with either of the two methods described above.

Figure 4 shows the field point pressures from the indirect BEM for BC1 and BC3 boundary conditions at a irregular frequency of 420 Hz. The sphere is made transparent to show the interior pressure fields for both the cases. The interior pressure field response is very high for the continuous velocity boundary condition case due to the presence of resonance at 420 Hz. Even though exterior pressure field exists it is relatively small compared to the interior resonance response. The amplitude of the pressure in the exterior field is small (O(10^2) Pa) compared to the amplitude of the pressure in the interior field (O(10^5) Pa) which affects the accuracy of the computed exterior sound field. It can be observed from Figure 4 that for the BC3 case the interior pressure field is largely suppressed resulting in accurate pressures in the exterior field. The relative difference between the interior and exterior pressure fields at the resonance frequency 420 Hz for both the continuous and discontinuous boundary conditions is shown more clearly in Figure 5, where the pressure variation along the radius from the center of the sphere is plotted. The analytical solution to the pulsating sphere radiation problem matches extremely well with the indirect BE solution for BC3 case.



(a) Continuous boundary condition (BC1)   (b) Discontinuous boundary condition (BC3)

Figure 4: Interior and exterior pressure fields (imaginary value) shown for a sphere with different types of boundary conditions at resonance frequency 420Hz. Note that in BC1 case, the interior response (O(10^5)) is much larger than the exterior response (O(10^2); whereas the interior reponse is suppressed in BC3 case.

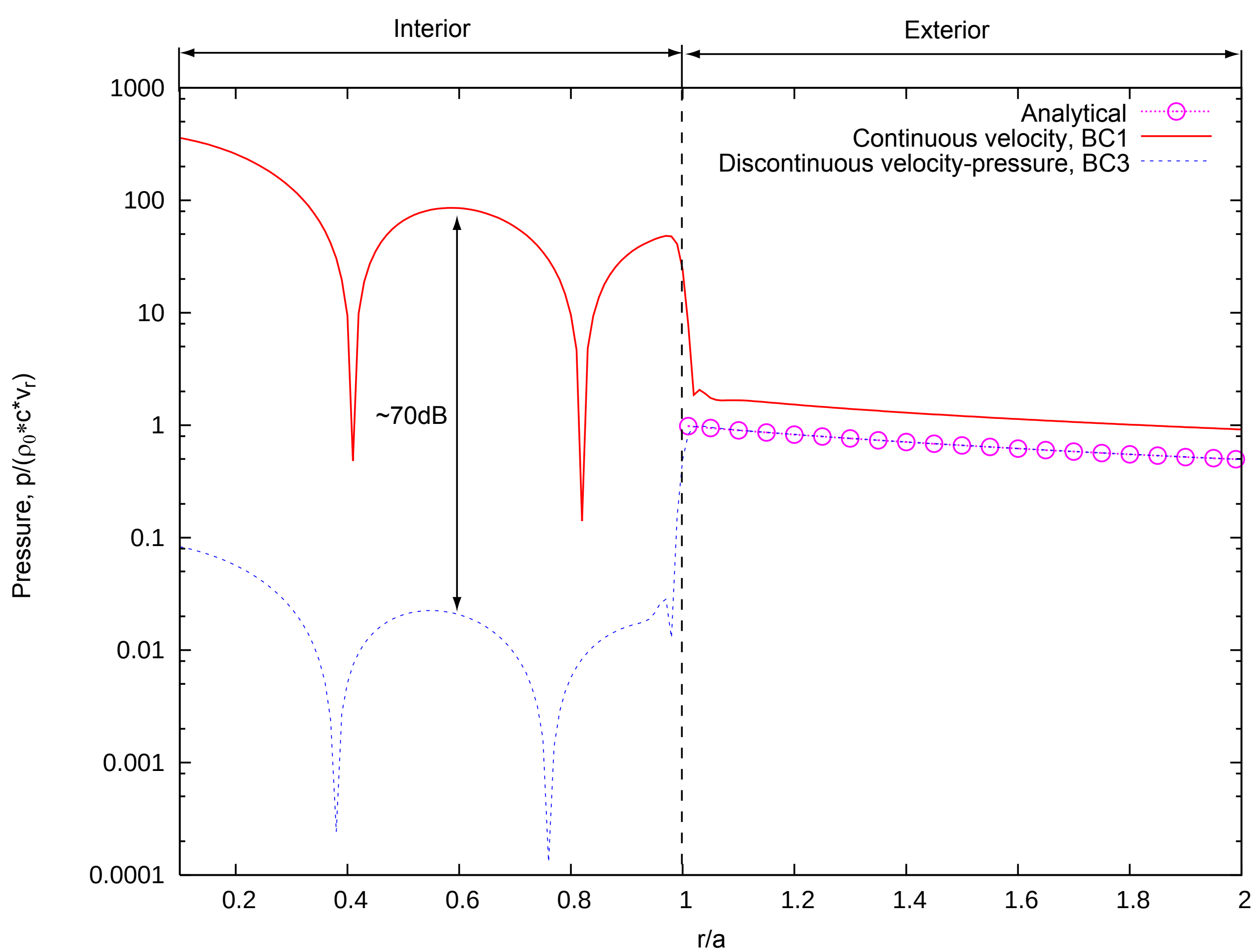


Figure 5: Pressure variation with distance from the center of a sphere for different boundary conditions at resonance frequency 420Hz.

4. CONCLUSIONS

The new indirect BE formulation presented in this paper handles different boundary condition types on each side of the boundary. The boundary conditions are represented by a general relation between pressure and pressure gradient on each side of the surface. Discontinuous boundary conditions are effectively used to address the non-uniqueness problem at irregular frequencies.

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