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Computing radiated sound power using quadratic power transfer function

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ABSTRACT

Pressure Acoustic Transfer Functions or Vectors (P-ATVs) relate the surface velocity of a structure to the sound pressure at a field point in the surrounding fluid. These functions depend on the structure geometry, properties of the fluid medium (sound speed and characteristic density), the excitation frequency and the location of the field point, but are independent of the surface velocity values themselves. Once the P-ATV is computed between a structure and a specified field point, we can compute the sound pressure at this point for any boundary velocity distribution by simply multiplying the P-ATV with the forcing function (surface velocity). These P-ATVs are usually computed by the application of the Reciprocity Principle.

In this work, we present a novel way to compute the Velocity Acoustic Transfer Vector (V-ATV) which is the relation between the surface velocity of the structure and fluid particle velocity at a field point. To our knowledge, the idea of the V-ATV and its computation is completely new and has not been published elsewhere.

By combining the P-ATVs and V-ATVs at a number of field points surrounding the structure, we obtain the Quadratic Power Transfer function (QPTF) which allows us to compute the sound power radiated by a structure for any surface velocity distribution. This allows rapid computation of the sound power for arbitrary surface velocity distributions and is useful in designing quiet structures by minimizing the radiated sound power.

1. INTRODUCTION

When designing quiet machines, it is often necessary to figure out the structure velocity distribution that minimizes the radiated sound power and then work backwards to see if this velocity distribution can be realized in operating conditions. The sound power calculation is *inside* the optimization loop.

A typical workflow is as follows:

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– For the given surface velocity distribution (v_n), compute the system of equations that relate the surface pressure to surface velocity (BEM system of equations).

– Solve these linear system of equations to determine the surface sound pressure (p) for the given surface velocity (v_n). As BEM system of equations are full, the corresponding influence matrices are not stored and iterative solvers are typically employed, along with acceleration techniques such as the Fast Multipole Method (FMM) for faster computation of the matrix-vector products.

– Integrate the average normal sound intensity $I_n = \frac{1}{2}\Re(p\tilde{v_n})$ over the surface of the radiator to compute the radiated sound power.

The time-consuming step in this workflow is the computation of the surface pressure (p) obtained by solving the boundary integral equations. Unfortunately, this step needs to be repeated once for every optimization iteration, thus slowing down the whole process. In this paper, we present a novel and *extremely fast approach* to compute the radiated sound power. Applying this technique will be very helpful, when designing structures to emit low noise.

The proposed method is *very general and works for all boundary condition types and any BEM formulation* (Helmholtz Integral Equation formulation, Burton Miller formulation etc, Indirect BEM variational formulation etc.). To convey the main idea without getting bogged down in the details, we will limit the discussion here to velocity boundary condition and the Helmholtz Integral Equation (HIE) formulation. Applying these principles, it is straight forward to derive the general equations for other boundary conditions and BEM formulations.

2. PRESSURE ACOUSTIC TRANSFER VECTOR (P-ATV)

2.1. Direct Problem

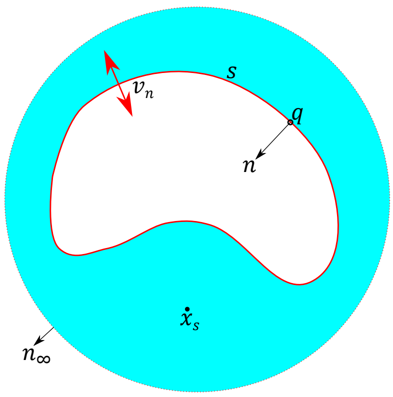


Figure 1: Direct Problem: Prescribed velocity boundary condition v_n on the structure. Sound pressure at the field point x_s is of interest.

Figure 1 shows the problem at hand. A structure with a bounding surface S is is submerged in a fluid and vibrating with a normal velocity of $v_n(q)$, where q is a boundary point. We are interested in computing $p(x_s)$, the sound pressure at the field point x_s.

The Helmholtz equation governs the sound pressure $p(x)$ variation in the fluid domain.

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

The boundary condition on surface S is

$$v_n = \overline{v}_n \tag{2}$$

where v_n is the normal component of velocity.

Instead of solving this problem directly as posed, we first compute the P-ATV between the structure and the field point x_s. Once the P-ATV is computed, obtaining the sound pressure at x_s for *any* v_n *involves just a multiplication* instead of solving the BEM system of equations. Of course, there is no free lunch. What are we losing? Solving the direct problem will enable us to compute the sound pressure at any point in the fluid, while multiplying the P-ATV with the boundary excitation will give us the sound pressure only at one particular field point x_s.

2.2. Reciprocity Principle

Reciprocity principle [1] holds true for *any linear system* and has wide application in mechanics, electromagnetics and acoustics. In simple terms, it states that if the positions of the source and the observer are interchanged, the measured response does not change.

Taking an example from mechanics, in the direct problem a force F_A is applied to a structure at location A and the deflection X_B is measured at point B. In the reciprocal problem, the force and response locations are interchanged, that is force F_B is applied at location B and response X_A is measured at location A. As per reciprocity principle:

$$\frac{X_B}{F_A} = \frac{X_A}{F_B} \tag{3}$$

If an equal force is used at both locations, then $X_B = X_A$, that is if the positions of source and observer are interchanged, the response value is identical. This is clearly illustrated in Figure 2 using a cantilever beam example.

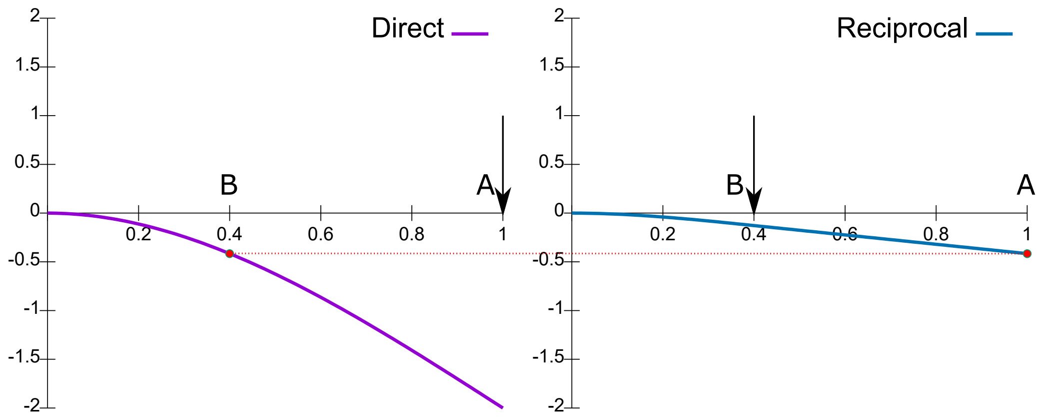


Figure 2: Illustration of the reciprocity principle using the example of a cantilever beam.

Generalizations of this principle to account for distributed loads on surfaces can be derived. In fact, the calculation of expressions for the pressure and velocity acoustic transfer functions is an application of this principle.

2.3. Computing P-ATV using Reciprocity

The Reciprocity principle is applied to compute the P-ATVs and its computation is well known as detailed in several publications [2,3] and US Patent 6,985,836 B2 [4].

Let S be the surface of the structure. In the reciprocal problem, a monopole source with a volume velocity ϖ is placed at the point of interest (x_s) and a acoustically rigid boundary condition ($v_n = 0$) is applied on the surface of the structure, as shown in Figure 3.

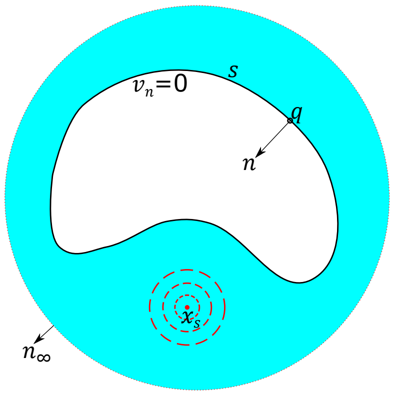


Figure 3: Reciprocal problem: Monopole source place at the location x_s. Acoustically hard boundary condition on the surface S.

The governing equation for the reciprocal problem is

$$\nabla^2\psi_p + k^2\psi_p = ikZ_0\,\varpi\,\delta(x - x_s) \tag{4}$$

The boundary condition on surface S is

$$\begin{aligned} v_n &= \frac{1}{ikZ_0}\frac{\partial\psi_p}{\partial n} = 0 \\ \frac{\partial\psi_p}{\partial n} &= 0 \end{aligned} \tag{5}$$

where ψ_p is the pressure, k is the wave number, $Z_0 = \rho_0 c$ is the characteristic impedance of the fluid medium, ϖ the monopole volume velocity and x_s the source location. We compute ψ_p on the surface S using BEM. P-ATV is related to the induced surface pressure ψ_p.

2.4. Green's Second Identity

Applying Green's second identity and using Sommerfeld's radiation condition on surface S_∞, we can write a relation between the two solutions p and ψ_p as

$$\int_V \left[\psi_p(\nabla^2 p + k^2 p) - p(\nabla^2\psi_p + k^2\psi_p)\right] dV = \int_S \left[\psi_p\frac{\partial p}{\partial n} - p\frac{\partial\psi_p}{\partial n}\right] dS \tag{6}$$

From boundary condition Equation 5, $\partial\psi_p/\partial n = 0$ on S,

$$\int_V \left[\psi_p(\nabla^2 p + k^2 p) - p(\nabla^2\psi_p + k^2\psi_p)\right] dV = \int_S \psi_p\frac{\partial p}{\partial n}\, dS \tag{7}$$

$\nabla^2 p + k^2 p = 0$ from Equation 1. Thus Equation 7 simplifies to

$$\int_V \left[-p(\nabla^2\psi_p + k^2\psi_p)\right] dV = \int_S \psi_p\frac{\partial p}{\partial n}\, dS \tag{8}$$

Using Equation 4, Equation 8 can be simplified as

$$\int_V -p\, ikZ_0\,\varpi\,\delta(x - x_s)\, dV = \int_S \psi_p\frac{\partial p}{\partial n}\, dS \tag{9}$$

Using the sifting property of the Dirac delta function, Equation 9 reduces to

$$-p(x_s)\, ikZ_0\, \varpi = \int_S \psi_p(q) \frac{\partial p}{\partial n}(q)\, dS(q) \tag{10}$$

Thus,

$$\begin{aligned} p(x_s) &= \frac{-1}{ikZ_0\, \varpi} \int_S \psi_p(q) \frac{\partial p}{\partial n}(q)\, dS(q) \\ &= \frac{-1}{\varpi} \int_S \psi_p(q) v_n(q)\, dS(q) \end{aligned} \tag{11}$$

Selecting volume velocity of the monopole source in the reciprocal problem as $\varpi = -1$ yields

$$p(x_s) = \int_S \psi_p(q) v_n(q)\, dS(q) \tag{12}$$

Thus $\psi_p(q)$, *the induced pressure on the radiator surface caused by a monopole source at the field point, is the Pressure Acoustic Transfer Function* (P-ATF). Discretized version of the P-ATF that incorporates the nodal pressures $\psi_p(q_i)$ and the nodal quadrature weights $w(q_i)$ is the Pressure Acoustic Transfer Vector (P-ATV).

$$\begin{aligned} p(x_s) &= \int_S \psi_p(q) v_n(q)\, dS(q) \\ &= \sum_{\text{nodes}\, q_i} \psi_p(q_i) v_n(q_i) w(q_i) \\ &= \begin{bmatrix} \psi_{p1}w_1 & \psi_{p2}w_2 & \ldots & \psi_{pi}w_i & \ldots & \psi_{pN}w_N \end{bmatrix} \begin{Bmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{ni} \\ \vdots \\ v_{nN} \end{Bmatrix} \\ p(x_s) &= [\text{P-ATV}(x_s)]\,\{v_n\} = \left[\hat{\psi}_p(x_s)\right]\{v_n\} \end{aligned} \tag{13}$$

The Pressure Acoustic Transfer Vector (P-ATV), also indicated by $\left[\hat{\psi}_p(x_s)\right]$ includes the nodal values of the induced pressure plus the integration weights. Inner product of $\left[\hat{\psi}_p(x_s)\right]$ with the vector of nodal normal velocities $\{v_n\}$ yields the pressure at the field point x_s.

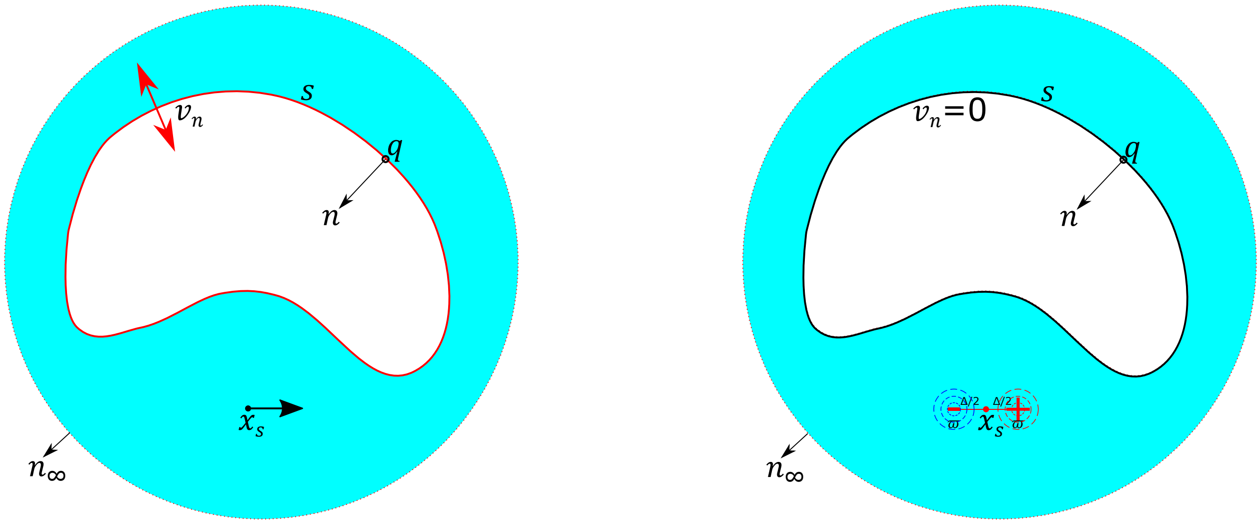
$$[\text{P-ATV}(x_s)] = \left[\hat{\psi}_p(x_s)\right] = \begin{bmatrix} \psi_p(q_1)w(q_1) & \psi_p(q_2)w(q_2) & \ldots & \psi_p(q_i)w(q_i) & \ldots & \psi_p(q_N)w(q_N) \end{bmatrix} \tag{14}$$

Equation 13 is used for computing $p(x_s)$, the sound pressure at the field point x_s for an arbitrary velocity v_n specified on the structure surface.

3. VELOCITY ACOUSTIC TRANSFER VECTOR (V-ATV)

To compute the Velocity Acoustic Transfer Function (V-ATF), we follow a procedure very similar to the one used P-ATF computation with one difference; Instead of using a point monopole source as was done for P-ATF, we will use a *dipole source at the field point* for the reciprocal problem. This allows us to extract the component of velocity in the *direction of the dipole*. If the full velocity vector at the field point is desired, we need to solve three separate reciprocal problems each with a different dipole orientation (x, y, or z), to extract all the Cartesian components of the velocity.

For simplicity we orient the dipole in the x-direction and obtain equations for computing the x-component of velocity $v_x(x_s)$.



(a) Direct problem with prescribed velocity boundary condition. v_x at field point x_s is desired.

(b) Reciprocal problem with a dipole source x_s and rigid boundary condition on S.

Figure 4: Direct and reciprocal problems for computing the Velocity-ATF.

3.1. Direct Problem

Direct problem with the prescribed velocity boundary condition $v_n = \bar{v}_n$ is shown in Figure 4(a). The Helmholtz equation governs the pressure field in the fluid surrounding the structure. It is desired to calculate the x-component of velocity at the field point x_s.

$$\nabla^2 p + k^2 p = 0 \tag{15}$$

The boundary condition on surface S is given as

$$v_n = \bar{v}_n \tag{16}$$

3.2. Reciprocal Problem

Let S be the surface of the structure. Consider an acoustic dipole of source strength ($D = \varpi\Delta$) located at the specified field point (x_s) and oriented along the x-direction, as shown in the Figure 4(b).

The governing equation has a forcing term due to the dipole source located at x_s.

$$\nabla^2 \psi_v + k^2 \psi_v = ikZ_0\, D\, \delta'(x - x_s)\delta(y - y_s)\delta(z - z_s) \tag{17}$$

The boundary S is considered acoustically rigid.

$$\frac{\partial \psi_v}{\partial n} = 0 \tag{18}$$

We compute ψ_v on the surface S using BEM.

3.3. Green's Second Identity

We derive an expression for the Velocity Acoustic Transfer Function (V-ATF) by combining the solutions to the direct and reciprocal problems using the Green's second identity.

$$\int_V \left[\psi_v(\nabla^2 p + k^2 p) - p(\nabla^2\psi_v + k^2\psi_v)\right] dV = \int_S \left[\psi_v \frac{\partial p}{\partial n} - p\frac{\partial \psi_v}{\partial n}\right] dS \tag{19}$$

From the governing Equation 15 and the rigid boundary condition from Equation 18, Equation 19 simplifies to

$$\int_V -p\left[ikZ_0\, D\,\delta'(x - x_s)\delta(y - y_s)\delta(z - z_s)\right] dV = \int_S \psi_v \frac{\partial p}{\partial n}\, dS \tag{20}$$

where

$$\delta'(x_s, y_s, z_s) = \lim_{\Delta\to 0} \frac{\delta(x_s + \frac{\Delta}{2}, y_s, z_s) - \delta(x_s - \frac{\Delta}{2}, y_s, z_s)}{\Delta} \tag{21}$$

Using the sifting property of the delta function, Equation 20 is reduced to

$$-ikZ_0 D \lim_{\Delta\to 0}\left[\frac{p(x_s + \frac{\Delta}{2}, y_s, z_s) - p(x_s - \frac{\Delta}{2}, y_s, z_s)}{\Delta}\right] = \int_S \psi_v \frac{\partial p}{\partial n}\, dS \tag{22}$$

$$-ikZ_0\, D\,\frac{\partial p}{\partial x} = \int_S \psi_v \frac{\partial p}{\partial n}\, dS \tag{23}$$

$$-ikZ_0\, D\,(ikZ_0 \frac{\partial v}{\partial x}) = \int_S \psi_v\,(ikZ_0\, v_n)\, dS \tag{24}$$

The component of velocity in the x-direction at the field point is given as

$$v_x(x_s, y_s, z_s) = -\frac{1}{D\,(ikZ_0)} \int_S \psi_v(q) v_n(q)\, dS\,(q) \tag{25}$$

Choosing the dipole source strength D to be $-1/ikZ_0$ yields the velocity acoustic transfer function (V-ATF) between the structure and the field point x_s.

$$v_x(x_s, y_s, z_s) = \int_S \psi_v(q) v_n(q)\, dS\,(q) \tag{26}$$

Physically, to a scalar multiple, the V-ATF is nothing but *the induced pressure on the structure surface S due to a dipole source with the proper orientation at the field point*. Discretized version of the V-ATF that incorporates the nodal values of the induced surface pressures $\psi_v(q_i)$ and the nodal quadrature weights $w(q_i)$ is the Velocity Acoustic Transfer Vector (V-ATV).

$$
\begin{aligned}
v_x(x_s, y_s, z_s) &= \int_S \psi_v(q) v_n(q)\, dS(q) \\
&= \sum_{\text{nodes } q_i} \psi_v(q_i) v_n(q_i) w(q_i) \\
&= \begin{bmatrix} \psi_{v1}w_1 & \psi_{v2}w_2 & \dots & \psi_{vi}w_i & \dots & \psi_{vN}w_N \end{bmatrix} \begin{Bmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{ni} \\ \vdots \\ v_{nN} \end{Bmatrix} \\
v_x(x_s, y_s, z_s) &= [\text{V-ATV}(x_s)]\,\{v_n\} = \left[\hat{\psi}_v(x_s)\right]\{v_n\} \qquad (27)
\end{aligned}
$$

The Velocity Acoustic Transfer Vector (V-ATV), also indicated by $\left[\hat{\psi}_v(x_s)\right]$ includes the nodal values of the induced pressure plus the integration weights. Inner product of $\left[\hat{\psi}_v(x_s)\right]$ with the vector of nodal normal velocities $\{v_n\}$ yields the velocity at the field point x_s.

4. QUADRATIC POWER TRANSFER FUNCTION

The sound power radiated by a structure is computed by integrating the time averaged normal sound intensity I_n over a measurement surface enclosing the radiator. Typically a spherical measurement surface is chosen for simplicity. The average sound intensity at a point x_f in the radial direction is given as

$$I_r(x_f) = \frac{1}{2}\Re[p(x_f)\tilde{v}_r(x_f)] \qquad (28)$$

where, $p(x_f)$ and $v_r(x_f)$ are the sound pressure and the radial component of velocity evaluated at the field point x_f, and $\sim$ represents complex conjugate operation.

If the pressure ATV $\left[\hat{\psi}_p(x_f)\right]$ and the radial velocity ATV $\left[\hat{\psi}_v(x_f)\right]$ between the structure and the field point x_f are precomputed and available, then the expression for average radial intensity at x_f can be written as

$$
\begin{aligned}
I_r(x_f) &= \frac{1}{2}\Re\left[\left(\left[\hat{\psi}_p(x_f)\right]\{v_n\}\right)\cdot conj\left(\left[\hat{\psi}_v(x_f)\right]\{v_n\}\right)\right] \\
&= \frac{1}{2}\Re\left[\left(\left[\hat{\psi}_p(x_f)\right]\{v_n\}\right)\cdot\left(\left[\tilde{\hat{\psi}}_v(x_f)\right]\{\tilde{v}_n\}\right)\right] \\
&= \frac{1}{2}\Re\left[[v_n]\left[\left\{\hat{\psi}_p(x_f)\right\}\otimes\left[\tilde{\hat{\psi}}_v(x_f)\right]\right]\{\tilde{v}_n\}\right] \qquad (29)
\end{aligned}
$$

$$I_{rj} = \frac{1}{2}\Re\left[[v_n]\left[\left\{\hat{\psi}_{pj}\right\}\otimes\left[\tilde{\hat{\psi}}_{vj}\right]\right]\{\tilde{v}_n\}\right] \qquad (30)$$

Equation 30 gives the time averaged radial intensity at a field point j as a quadratic function of surface normal velocity vector $\{v_n\}$. The highlighted term is the *outer product* between pressure ATV and the conjugate of the radial velocity ATV at the field point j, and is a matrix of rank one.

The radiated sound power Π_{av} is the weighted sum of time averaged radial intensity at the field points and is given as:

$$\Pi_{\text{av}} = \sum_j I_{rj}w_j = \frac{1}{2}\Re\,[v_n]\sum_j\left[w_j\left\{\hat{\psi}_{pj}\right\}\otimes\left[\tilde{\hat{\psi}}_{vj}\right]\right]\{\tilde{v}_n\} = \frac{1}{2}\Re\,[v_n]\,[\Omega]\,\{\tilde{v}_n\} \qquad (31)$$

where the **Resistance Matrix** $[\Omega]$ is given as

$$[\Omega] = \sum_j w_j \left\{\hat{\psi}_{pj}\right\} \otimes \left[\tilde{\hat{\psi}}_{vj}\right] \tag{32}$$

and sound power Π_{av} is

$$\Pi_{\text{av}} = \frac{1}{2} \Re \, [v_n] \, [\Omega] \, \{\tilde{v}_n\} \tag{33}$$

Thus, combining the pressure ATV and the radial velocity ATV's at a number of field points on a measurement mesh, we obtain the quadratic sound power transfer function shown in Equation 33. This is a wonderful result – *knowing only the surface velocity of the structure* we can now quickly compute the radiated sound power without any intermediate calculations.

The concept of a resistance matrix for sound power was explored in Koopman et. al [5]. However, it was limited to planar radiators, and small sources (low frequency approximation). The derivation here is completely general.

An interesting observation from Equation 32 is that the resistance matrix $[\Omega]$ is expressed as a *sum of several rank one matrices*, much like an onion with its layered shells. We will explore in future work, what the physical significance of this decomposition is; here we just point out this interesting fact. Resistance matrix $[\Omega]$ being a sum of many rank one matrices opens the possibility of storing just the P-ATV and V-ATVs for each frequency of interest (Memory=$2NM$, where N is the number of nodes and M is the number of field points) instead of storing the entire resistance matrix (Memory=N^2) for each frequency - very efficient as the number of nodes N far exceeds the number of field points M.

5. EXAMPLE

An example of an automotive transfer case is considered. At a frequency of 1000 Hz, the pressure and velocity ATVs between the transfer case and a field point below it, at location $x_s(0.21, -0.46, -0.14)$ are computed by solving the corresponding reciprocal problems using *Coustyx* from ANSOL [6, 7].

5.1. Pressure ATV

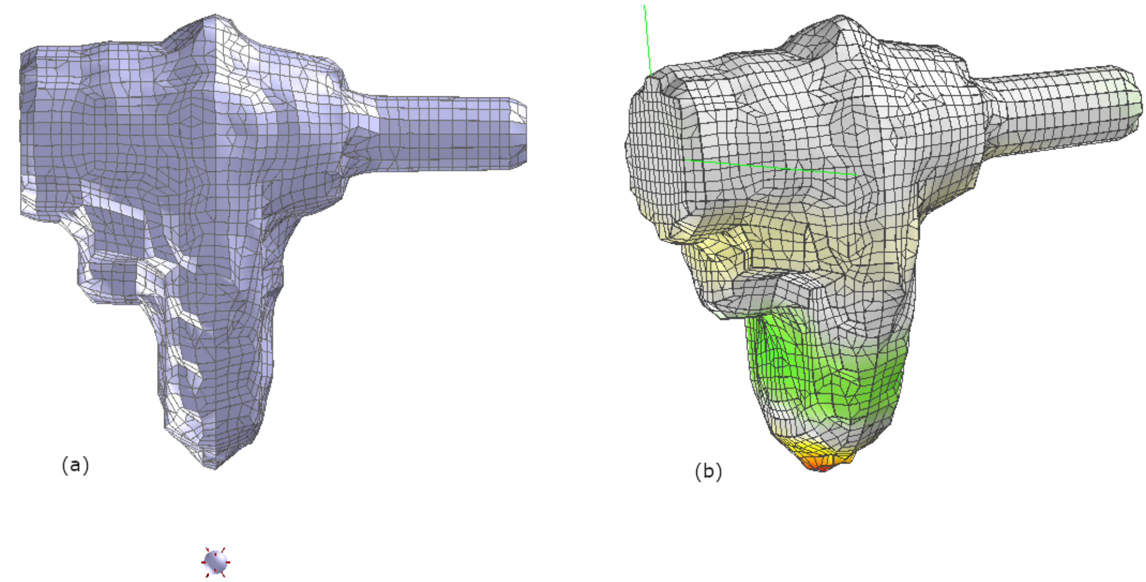


Figure 5: (a) Problem set up for computing P-ATV. Reciprocal problem with a monopole source at x_s. (b) P-ATV between x_s and the transfer case at 1000 Hz.

From Figure 5, it is clear the portions of the surface that are closer to the chosen field point x_s will have a larger contribution to the pressure at x_s.

5.2. Velocity ATV

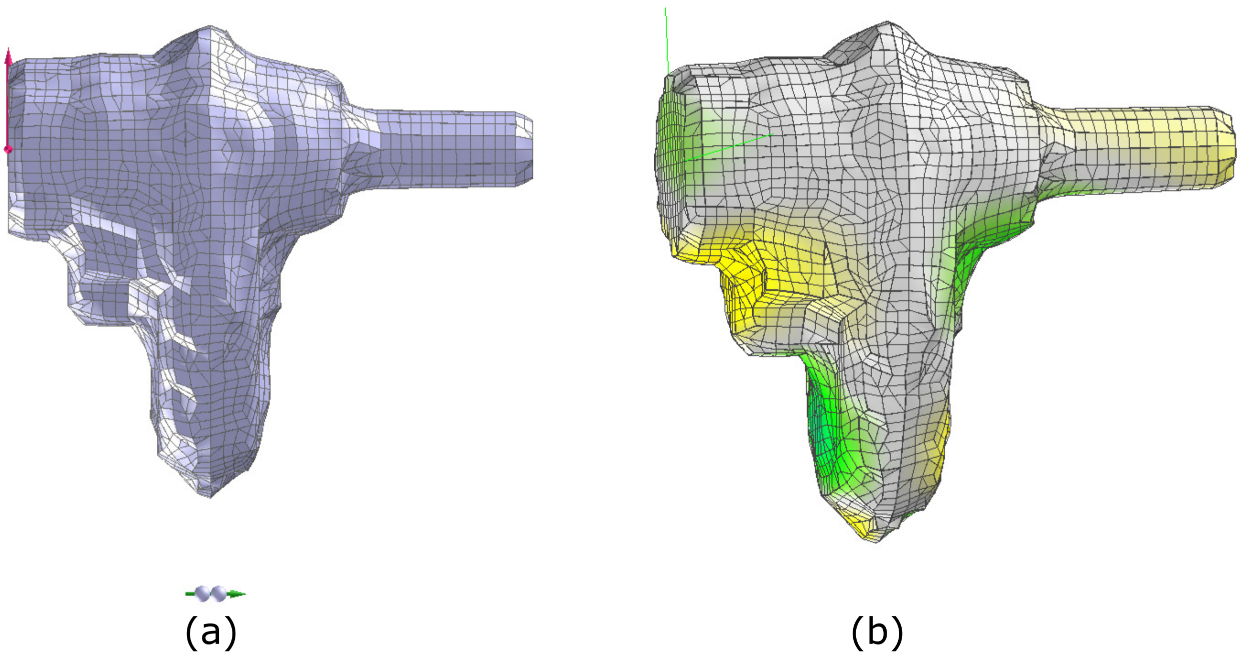


Figure 6: (a) Problem set up for computing V-ATV. Reciprocal problem with a dipole source at x_s. (b) V-ATV between x_s and the transfer case at 1000 Hz.

From Figure 6, it is seen that the character of the V-ATV is quite different than the P-ATV. That is, the surface portions contributing to acoustic particle velocity at x_s are very different than the one contributing to the pressure.

6. CONCLUSION

This paper builds upon the idea of the pressure ATVs $\{\hat{\psi}_p\}$ and develops *completely new* formulations for constructing the velocity ATVs $\{\hat{\psi}_v\}$ and the sound power resistance matrix $[\Omega]$. These transfer vectors and matrices are very useful for fast computation of pressure and velocity at a field point, and radiated sound power.

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